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FRACTIONAL MATCHINGS AND COVERS IN INFINITE HYPERGRAPHS

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A strong version of the duality theorem of linear programming is proved for fractional covers and matchings in countable graphs. It is conjectured to hold for general hypergraphs. In Section 2 we show that in countable hypergraphs there does not necessarily exist a maximal matchable set, contrary to the situation in graphs.

1. Fractional matchings and covers

Let $H=(V,E)$ be a hypergraph (finite or infinite). A *fractional cover* of H is a function $f\colon V\to R$ satisfying $\sum\{f(v)\colon v\in e\}\geqq 1$ for every $e\in E$. A *fractional matching* is a function $g\colon E\to R$ satisfying $\sum\{g(e)\colon v\in e\}\leqq 1$ for every $v\in V$. If the range of f (or g) is contained in $\{0, 1\}$ it is called simply a *cover* (or *matching*, respectively), and then we refer also to the set $\{v\colon f(v)=1\}$ as a cover and to the set $\{e\colon g(e)=1\}$ as a matching.

Given a fractional cover f and a fractional matching g there holds:

(1) $$\sum_{v\in V} f(v) \geqq \sum_{e\in E}\sum_{v\in e} f(v)\,g(e) \geqq \sum_{e\in E} g(e).$$

For finite H, by the duality theorem of linear programming there exist a fractional cover f and a fractional matching g such that

(2) $$\sum_{v\in V} f(v) = \sum_{e\in E} g(e).$$

By (1) this implies the so called "complementary slackness conditions":

(3a) $$f(v)\neq 0 \quad \text{implies} \quad \sum_{v\in e} g(e) = 1$$

(3b) $$g(e)\neq 0 \quad \text{implies} \quad \sum_{v\in e} f(v) = 1.$$

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Conditions (2) and (3) are equivalent for finite hypergraphs, but not for infinite hypergraphs. While (2) can easily be shown for any hypergraph, we cannot prove the stronger condition (3), and we state it as a conjecture.

Conjecture. *In any hypergraph there exist a fractional cover f and a fractional matching g satisfying* (3).

Here we prove the conjecture for countable graphs, in a slightly stronger form. For finite graphs this is known as the "two-matching theorem" (see [1]).

Theorem. *Let $G=(V,E)$ be a countable graph. There exist a fractional cover $f\colon V\to\{0, 1/2, 1\}$ and a fractional matching $g\colon E\to\{0, 1/2, 1\}$ satisfying* (3).

For the proof of the theorem we need a few preliminaries. Given any set A, a subset K of $A\times A$ (unordered pairs), an element a of A and a subset B of A, we denote by $K\langle a\rangle$ the set $\{x\in A\colon \{a,x\}\in K\}$, by $K(a)$ the only element of $K\langle a\rangle$ if $|K\langle a\rangle|=1$, and by $K[B]$ the set $\bigcup\{K\langle b\rangle\colon b\in B\}$. A *matching* in a graph $G=(V,E)$ is a subset I of E such that $|I\langle v\rangle|\leqq 1$ for every $v\in V$. We write $s(I)=I[V]$. For $X\subseteq V$ let $I\restriction X=I\cap(X\times V)$. A *cover* is a subset C of V such that $E\subseteq C\times V$. (Note that this agrees with earlier definitions).

Let $\Gamma=(W,F)$ be a bipartite graph with bipartition $W=P\cup Q$. A subset C of P is called *critical* if there exist a matching I in Γ with $s(I)\supseteq C$, but for every matching I such that $s(I)\supseteq C$ and $I\subseteq C\times V$ there holds $I[C]=F[C]$. The following is a basic fact about matchability in countable bipartite graphs:

Theorem PS [2]. *If $\Gamma=(W,F)$ is countable, bipartite with bipartition $W=P\cup Q$ then there exists a matching I with $P\subseteq s(I)$ if and only if there does not exist a critical subset C of P and an element a of $P\setminus C$ such that $F\langle a\rangle\subseteq F[C]$.* ∎

Proof of Theorem. Form a bipartite graph $\Gamma=(V'\cup V'', F)$ where $V'=\{v'\colon v\in V\}$, $V''=\{v''\colon v\in V\}$, and $F=\{\{u',v''\}\colon \{u,v\}\in E\}$. For any subset W of V write $W'=\{w'\colon w\in W\}$, $W''=\{w''\colon w\in W\}$. Let C' be a maximal critical subset of V' (by containment). It is easy to see, by Zorn's Lemma, that a maximal critical set always exists (see [2, Lemma 1]). Let I be a matching of C' and write $X''=I[C']=F[C']$. Let $B'=\{v'\in V'\setminus C'\colon F\langle v'\rangle\subseteq X''\}$, $A'=V'\setminus B'\setminus C'$ and let $\bar{\Gamma}$ be the subgraph of Γ spanned by $A'\cup(V''\setminus X'')$. By the maximality of C' the side A' contains no nonempty critical set in $\bar{\Gamma}$, and by the definition of A' for no $a'\in A'$ does there hold $F\langle a'\rangle=\emptyset$. Hence, by Theorem PS there exists a matching J in $\bar{\Gamma}$ with $s(J)\supseteq A'$. We can regard J as a matching in Γ. By the criticality of C' and by the definitions of A' and B' the set $A'\cup X''$ is a cover in Γ.

Lemma. *If $v_0''\in X''$ and $I(v_0'')=v_1'$ and $v_1''\in X''$ then $v_0'\in C'$.*

Proof. Suppose that $v_0'\notin C'$. Let $v_2'=I(v_1'')$. Since $v_0'\notin C'$ we have $v_2'\neq v_0'$. Since $\{v_2',v_1''\}\in F$ also $\{v_1',v_2''\}\in F$, and since $A'\cup X''$ is a cover in Γ and $v_1'\notin A'$ it follows that $v_2''\in X''$. Thus $v_3'=I(v_2'')$ is defined. By the same argument as above $v_3''\in X''$ and $v_4'=I(v_3'')$ is defined and $v_4'\neq v_0'$. Continuing in this way we obtain a sequence v_i, $0\leqq i<\omega$ of distinct vertices such that $v_{i+1}'=I(v_i'')$ and $v_i''\in X''$ for every $i\geqq 0$. Let $H=I\setminus\{\{v_i'',v_{i+1}'\}\colon 0\leqq i<\omega\}\cup\{\{v_i',v_{i+1}''\}\colon 0\leqq i<\omega\}$. Clearly H is a matching, $s(H)\supseteq C'$ and $v_0''\in F[C']\setminus H[C']$. This contradicts the criticality of C'. ∎

Let $T_1=\{x\in V: x''\in X''$ and $x'\notin C'\}$. Let $v_0\in T_1$ and define $v_1'=I(v_0'')$. By the Lemma $v_1''\notin X''$. Since $A'\cup X''$ is a cover in Γ and $\{v_0', v_1''\}\in F$ it follows that $v_0'\in A'$. Assume that $v_1''\in s(J)$, and let $v_2'=J(v_1'')$. Then $v_2'\in A'$. Since $\{v_1', v_2''\}\in F$ and $A'\cup X''$ is a cover in Γ and $v_1'\notin A'$ it follows that $v_2''\in X''$. Let $v_3'=I(v_2'')$. By the Lemma $v_3''\notin X''$, and if $v_3''\in s(J)$ then $v_4'=J(v_3'')$ satisfies, by the same argument as above, $v_4''\in X''$. In this way a sequence v_i, $0\leqq i$, is generated, such that $v_{2k}'\in A'$, $v_{2k}''\in X''$, $v_{2k+1}'=I(v_{2k}'')$ and $v_{2k}'=J(v_{2k-1}'')$. Suppose, if possible, that this sequence is infinite. Define $I^*(v_{2k+1}')=v_{2k+2}''$ for $k=0, 1, \dots$ and $I^*(x')=$ $=I(x')$ for any other $x'\in C'$. Then I^* is a matching satisfying $s(I^*)\supseteq C'$ and $v_0''\in F[C']\setminus I^*[C']$, contradicting the criticality of C'.

Thus the sequence v_i is finite. Writing w with $w''=J(v_0')$, one case in which this could possibly occur is that $J(v_{2k+1}'')=w'$ for some $k\geqq 0$. But then $\{v_{2k+1}', w''\}\in F$, and since $v_{2k+1}'\notin A'$ and $w''\notin X''$ this contradicts the fact that $A'\cup X''$ is a cover. The only possibility remaining is that $v_{2k+1}''\notin s(J)$ for some $k\geqq 0$. Define then $S_1(v_0)=\{v_{2i}: 0\leqq i\leqq k\}$ and

$$\tilde{J}(v_{2i}') = v_{2i+1}'', \quad (0 \leqq i \leqq k). \tag{4}$$

Let $T_2=\{x\in V: x'\in A'$ and $x''\notin s(I\cup J)\}$. Let $v_0\in T_2$ and let $v_1''=J(v_0')$. Since $\{v_1', v_0''\}\in F$ and $A'\cup X''$ is a cover in Γ and $v_0''\notin X''$ it follows that $v_1'\in A'$. Let $v_2''=J(v_1')$. By the same argument $v_2'\in A'$, and thus $v_3''=J(v_2')$ is defined. In this way a sequence v_i, $0\leqq i<\omega$ is generated such that $v_{i+1}''=J(v_i')$ for any $0\leqq i<\omega$. Define $S_2(v_0)=\{v_{2i+1}: 0\leqq i<\omega\}$ and $\tilde{J}(v_{2i+1}')=v_{2i}''$ for every $0\leqq i<\omega$.

Let now

$$M = I\cup J\setminus(J|(\cup\{S_1(v):\ v\in T_1\}\cup\cup\{S_2(v):\ v\in T_2\})\cup$$
$$\cup\{\{u,\tilde{J}(u)\}:\ u\in S_1(v),\ v\in T_1 \quad\text{or}\quad u\in S_2(v),\ v\in T_2\}$$

(i.e. M is obtained from $I\cup J$ by replacing $\{u, J(u)\}$ by $\{u, \tilde{J}(u)\}$ whenever $\tilde{J}(u)$ is defined). It is easy to check that M is still a matching in Γ.

Define h: $V'\cup V''\to\{0, 1/2\}$ by $h(t)=1/2$ if $t\in A'\cup X''$, $h(t)=0$ otherwise. Also define l: $F\to\{0, 1/2\}$ by $l(f)=1/2$ if $f\in M$, $l(F)=0$ otherwise. Let now f: $V\to\{0, 1/2, 1\}$ and g: $E\to\{0, 1/2, 1\}$ be defined by $f(v)=h(v')+h(v'')$ and $g(\{u, v\})=l(\{u', v''\})+l(\{u'', v'\})$. If $e=\{u, v\}\in E$ then both $\{u', v''\}$ and $\{u'', v'\}$ are in F, and hence at least one of u', v'' is in the cover $A'\cup X''$ of Γ, and so is at least one of u'', v'. Thus at least two of u', u'', v', v'' are in $A'\cup X''$, hence $f(u)+f(v)=$ $=h(u')+h(u'')+h(v')+h(v'')\geqq 1$, showing that f is a fractional cover. If $v\in V$ then

$$\sum_{v\in e} g(e) = \sum_{v'\in f} l(f)+\sum_{v''\in f} l(f) = \sum_{v'\in f\in M}\frac{1}{2}+\sum_{v''\in f\in M}\frac{1}{2},$$

and since M is a matching in Γ we have $\sum\{g(e):\ v\in e\}\leqq 1$, showing that g is a fractional matching.

Let us show that f and g satisfy (3). Suppose that $f(v)\neq 0$. Then either $v'\in A'$ or $v''\in X''$. Consider first the case $v'\in A'$. Let $M(v')=u''$. Then $\{u', v''\}\in F$ and hence either $u'\in A'$ or $v''\in X''$. If $v''\in X''$ then both v' and v'' belong to $s(M)$.

implying $\sum\{g(e)\colon v\in e\}=1$, which is (3a). If $v''\notin X''$ and $u'\in A'$ then by the definition of T_2 and $\tilde{J}$ there holds $v''\in s(\tilde{J})\subseteq s(M)$ again proving (3a).

Assume now that $v''\in X''$. Let $u'=I(v'')$. Since $\{v',u''\}\in F$ either $u''\in X''$ or $v'\in A'$. If $u''\in X''$ then, by the Lemma, $v'\in C'$, hence $v'\in s(M)$, and since also $v''\in s(M)$ there holds $\sum\{g(e)\colon v\in e\}=1$, i.e. (3a). If $v'\in A'$ then $v'\in s(M)$, again implying (3a).

To show (3b) let $e=\{u,v\}\in E$ be such that $g(e)\neq 0$. Then either $\{u',v''\}\in M$ or $\{v',u''\}\in M$, and by symmetry we may assume the first. Suppose first that $\{u',v''\}\in I$, so that $v''\in X''$. If $\sum\{f(v)\colon v\in e\}>1$ then there must hold: $u''\in X''$ and $v'\in A'$. But this is impossible, by the Lemma. Assume now that $\{u',v''\}\in M\setminus I$. Then $u'\in A'$. If $\sum\{f(v)\colon v\in e\}>1$ then $u''\in X''$ and $v'\in A'$. But then $u\in T_1$, and by (4) it follows that $\tilde{J}(u')\neq v''$, contrary to the assumption that $\{u',v''\}\in M$. We have thus shown that $\sum\{f(v)\colon v\in e\}=1$, which is (3b).

2. Maximal matchable sets in countable hypergraphs

This seems as an appropriate place to point out a dissimilarity between countable graphs and countable hypergraphs, concerning matchability. A set X in a hypergraph is *matchable* if $X=\bigcup M$ for some matching M. Steffens [3] showed that in any countable graph there exists a maximal matchable set. The following example shows that this does not hold in general for hypergraphs, even if the size of the edges is finitely bounded. Let A, B, C, D be disjoint sets such that $|A|=|B|=|C|=|D|=\aleph_0$. Let $f\colon A\to D$ and $g\colon B\to C$ be bijections. Let $H=(V,E)$ be defined by $V=A\cup B\cup C\cup D$ and

$$E=\big\{\{a,f(a)\}\colon a\in A\big\}\cup\big\{\{a,c,d\}\colon a\in A,\ c\in C,\ d\in D\big\}\cup$$
$$\cup\big\{\{b,g(b),d\}\colon b\in B,\ d\in D\big\}.$$

It is easy to check the following:

(a) If $X\subseteq V$ is matchable then there exists a matchable set Y such that $Y\supseteq X\cup C\cup D$.

(b) If $Y\supseteq C\cup D$ then it is matchable if and only if $|Y\setminus(A\cup B)|=\aleph_0$.

But there does not exist a maximal set among those containing $C\cup D$ and omitting countably many vertices from $A\cup B$. Hence H does not contain a maximal matchable set.

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